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Self-Preservation in Fully Expanded Turbulent Coflowing Jets

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Introduction

THE problem of the flow in a round jet, when the surrounding air is at rest, has been fully investigated by many authors.[1,2] The present work analyzes more recent data and finds that correlations suggested by Boynton are valid for a wide range of experiments. Furthermore, the correlations are extended to coflowing jets, such as those existing in a low altitude exhaust plume of an ascending missile. The general situation of interest considers axisymmetric turbulent mixing between an inner stream of fluid injected parallel to a moving outer stream.

Jet Mixing with Stagnant Air

The classical analytical treatment[1,2] of jet mixing considers three main regions as shown in Fig. 1. These consist of 1) potential core, 2) the transitional or developing region, and 3) the similarity region, where suitably scaled profiles are self-preserving with axial distance.

It is well established that radial velocity profiles obtained at different axial locations in the fully developed region of subsonic, constant density, axially symmetric, and turbulent free jets issuing into a stagnant medium can be normalized to congruent curves by plotting them in the proper reduced coordinates.[3]

$$(U_{(r)}/U_C) \propto f_1(r/x)^2$$

Where $U_{(r)}$ is the velocity measured at a given radial distance (r) from the centerline, U_C is the centerline velocity and x is the axial distance along the centerline. All points obtained downstream a certain axial distance from the origin, i.e., in the similarity region, fall on a universal curve. Such self-preserving flow cannot exist in the "real" coordinate system where significant density variations occur in the flowfield. Furthermore, it has been observed experimentally that supersonic jets tend to spread less than low speed jets. Low speed variable density flows are self-preserving in the Howarth-Dorodnitsyn coordinates, where a transformation is performed on the radial distance.

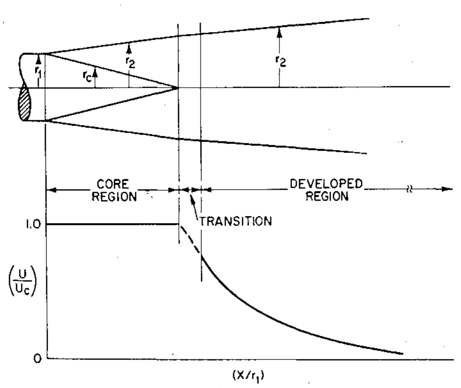

Fig. 1 Idealized jet structure.

$$R = \left[2\int_0^r \left(\frac{\rho}{\rho_\infty}\right) r'dr'\right]^{1/2}$$

Where ρ is the density at a point and ρ_∞ is the ambient density. This transformation[4] converts the momentum and continuity equations to their incompressible form. Furthermore, it is necessary to postulate that the supersonic jets can be scaled using a spreading constant, which is a function of the Mach number. It has been found[5] that their scaling constant σ represents the square root of the ratio of the initial total enthalpies to the initial static enthalpies.

$$[\sigma = h_i^0/h_i]^{1/2} = [1 + 1/2(\gamma - 1)M_i^2]^{1/2}$$

A similarity analysis applied[5] to the transformed momentum equation leads to the conclusion that in the fully developed region

$$(U_C/U_i) = (\rho_\infty/\rho_i)^{-1/2} I^{-1/2} \sigma r_i / x$$

where I is the momentum flux defined by

$$I = 2\int_0^\infty f_2{}^2(\eta)\eta d\eta$$

$\eta = R\sigma/x$ and $f_2(\eta)$ is the universal function represented by the low speed data, and subscript i, means the initial conditions (jet nozzle exit). Furthermore, it is assumed that the momentum flux is the same as for the low speed incompressible flows[1] i.e. $I = \frac{1}{4}$.

Data from several sources have been studied by Boynton.[5] This present study applies these findings to additional high and low speed, hot and cold experimental flows.[6] Correlation has been found to be excellent and is given in Fig. 2.

Effect of the Moving External Stream

It has been previously shown[7] for free jets mixing with a moving ambient environment that the excess dimensionless velocity profile (that velocity over the unperturbed external flow) is the same universal function as for the case of a stagnant environment that is in the developed region

$$(U - U_\infty)/(U_C - U_\infty) = \exp[-0.692(r/r_2)^2]$$

where r_2 = position where $U = \frac{1}{2}U_C$.

Theoretical studies were reported in Ref. 8 but their result is inconvenient to apply. Experimental studies of the decay of a jet exhausting into a moving coaxial stream were performed in Ref. 7 by using streams of unequal compositions. It was noted that the presence of a moving external stream

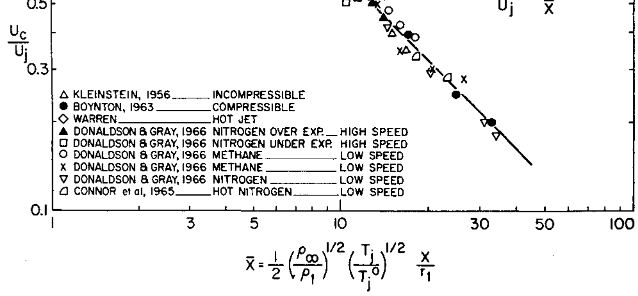

Fig. 2 Decay of centerline velocity.

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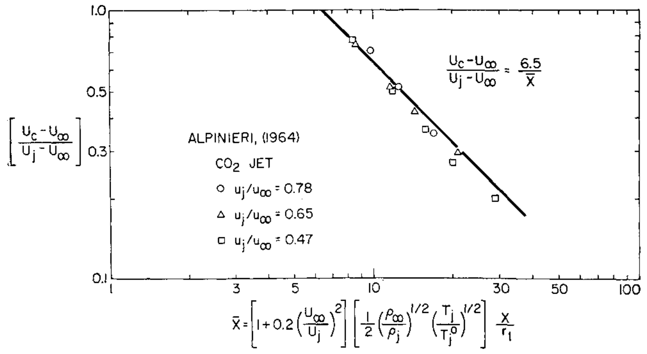

Fig. 3 Co-axial jet centerline velocity decay.

caused a decrease in the rate of spread of the central jet and the rate of growth of the half radius was no longer linear, furthermore, it was found that

$$(U_C - U_\infty)/(U_i - U_\infty) \propto 1/x$$

This result indicates that there is a possibility to define the proportionality constant such that the flow will be self-preserving. Applying the transformations reported for the stagnant external conditions did not yield the necessary transformation. It is necessary, also, to postulate that further transformation is possible, and that the proportionality constant is a function of the external fluid velocity such that $\beta = 1$ when $U_\infty = 0$. By analyzing the data reported in Ref. 9 we obtain $\beta = 1 + 0.2(U_\infty/U_i)^2$. Combining this result with Boynton's correlation we obtain an empirical result with good approximation.

$$(U_C - U_\infty)/(U_i - U_\infty) = 2(\rho_\infty/\rho_i)^{-1/2}\beta^{-1}\sigma r_i/x$$

Figure 3 shows the centerline velocity vs the normalized coordinate.

Conclusion

The results of this study imply that the Howarth transformation may be employed to reduce all fully developed jet velocity data to a common scale if a spreading constant that is a function of the Mach number is employed. Furthermore, applying a second spreading constant which is a function of the external flow condition it is possible to reduce data obtained in all coaxially flowing jets to a common scale. The significance of this finding cannot be assessed without more data.

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Comments on the Convergence of Finite Element Solutions

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I. Introduction

WITHIN the last few years, many developments have been made in the basic finite element. New trends toward curved elements to analyze curved structures have forced the profession to look closer at some basic concepts of the displacement function assumed for the element. This note deals with rigid-body mode shapes in displacement functions and their effect on the rate of convergence. Also considered is the effect of two shell theories and two load lumping procedures.

Some researchers[1,2] have included rigid-body mode shapes in their displacement functions for curved elements, but others[3] still insist they are not needed. There are researchers who do not include the rigid-body mode shapes, but correct the final displacement function or stiffness matrix, using equilibrium considerations.[4]

Cantin and Clough[2] point out that, for flat elements, the polynomial is the best displacement function assumption; but when a curved element is used, these polynomials cannot account for rigid-body motion. Therefore, if a structure, which has substantial rigid-body motion, is analyzed with these elements, convergence to a satisfactory solution is not always assured.

Walz, Fulton, Cyrus, and Eppink[5] show analytically that there is a large error term present in the curved element with the polynomial displacement function which does not have the provisions for unstrained rigid-body motion. They advance the idea that this large error could be due to the omission of the rigid-body mode shape. Of course, if when the mesh size is reduced, the rigid-body modes are recovered, slow convergence to a satisfactory result is possible. For this fine a mesh the flat element is equally as good.

The objective of this Note is to examine in greater detail the convergence for four different finite elements when used to analyze the circular arch presented in Ref. 5.

II. Elements

The standard straight and several curved elements were compared with each other with respect to their convergence to a satisfactory solution to the displacement of the example structure.

A. Straight element

The straight element stiffness matrix was derived using the following displacement function:

$$u = a_1 + a_2x \tag{1}$$

$$w = a_3 + a_4x + a_5x^2 + a_6x^3 \tag{2}$$

where u and w are the x and y displacements and the a's are generalized coordinates. This displacement function includes rigid-body mode shapes for the flat element structure and converges to an exact solution as shown in Ref. 5.

B. Curved elements

In all the curved elements, the displacement function has the following form:

$$u = f(s) \tag{3}$$

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